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LETTER TO THE EDITOR

Non-local symmetries and a working algorithm to isolate integrable geometries

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Abstract. A 1-parameter group of non-local symmetries of non-homogeneous, nonlinear Schrödinger system with the coupling function linear in x is presented. On using this symmetry we are able to insert the spectral parameter into the non-parametric Lax pair in all cases which are known to be integrable. A working algorithm to isolate integrable geometries, proposed by Levi, Sym and Tu, is extended.

The classical geometry of surfaces is an inexhaustible source of systems of nonlinear partial differential equations. They are known as reduced Gauss–Mainardi–Codazzi (RGMC) equations and describe immersions of surfaces in ambient spaces [1, 2, 5].

Especially interesting are so called *integrable geometries* [3], i.e. classes of surfaces defined by RGMC equations which are integrable in the sense of the theory of solitons [4]. For instance the remarkable properties of pseudospherical surfaces (see [2]) can be explained by the fact that the corresponding RGMC equations are equivalent to the Sine–Gordon equation which is one of the 'classical' soliton systems [4]. A large class of integrable (or soliton) systems admits an analogical geometrical interpretation in the framework of the theory of soliton surfaces [5].

Recently there was proposed a simple working algorithm to isolate integrable geometries [6]. This algorithm is based on the fact that RGMC equations can always be interpreted as integrability conditions for an overdetermined system of linear partial differential equations (Gauss–Weingarten equations [1, 2]). On the other hand, soliton systems are integrability conditions for a linear system (known as a Lax pair) which contains the so called *spectral parameter* [4]. Therefore RGMC equations can be considered as a reasonable candidate for integrable (soliton) systems if one is able to insert a free parameter into GW equations in such a way that their integrability conditions are left unchanged.

Sometimes the spectral parameter can be introduced by a scale transformation [7, 8]. The working algorithm of [6] generalizes these observations: the free parameter is introduced by Lie point symmetries. The algebra of point symmetries of RGMC equations is computed in the standard way [9]. Then, by solving a system of ordinary differential equations, one obtains corresponding 1-parameter groups of point transformations. Finally, one studies the action of each 1-parameter group on the GW equations. In some cases this action inserts a free parameter which cannot be gauged out. The conjecture has been made [6, 10] that then RGMC equations are integrable in the sense of the theory of solitons.

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The presented algorithm has been successfully applied to the surfaces of (non-constant) negative Gaussian curvature: the Lax pair for the so called Bianchi system was found [11].

Presumably, the conjecture of [6] is true, i.e. Lie point symmetries are one more working tool to isolate integrable systems. A separate question is: does this algorithm isolate *all* integrable cases of RGMC equations?

Unfortunately, the answer is negative. Indeed, let us consider the so called non-homogeneous, nonlinear Schrödinger (NHNS) system [12]:

$$iq_{,t}+(fq)_{,xx}+2qR=0 \tag{1a}$$

$$R_{,x}-(f|q|^2)_{,x}-f_{,x}|q|^2=0 \tag{1b}$$

where a comma denotes differentiation, $q=q(x,t)\in\mathbb{C}$ and $R=R(x,t)\in\mathbb{R}$ are dependent variables, and $f=f(x,t)$ is a given real function. This system is closely related to the inhomogeneous Heisenberg ferromagnet equation [12–16] and in this context f is called a 'coupling function'.

The NHNS system has a geometrical interpretation being equivalent to RGMC equations for surfaces endowed with semi-geodesic coordinates (x,t) [14–17]. The corresponding GW equations can be easily transformed into the following simple form [6, 17]:

$$\Psi_{,x}=\begin{pmatrix}0 & q\\ -\bar{q} & 0\end{pmatrix}\Psi \tag{2a}$$

$$\Psi_{,t}=\begin{pmatrix}iR & i(qf)_{,x}\\ i(\bar{q}f)_{,x} & -iR\end{pmatrix}\Psi \tag{2b}$$

where the 'bar' denotes complex conjugate.

The integrability conditions for the linear system (2) are equivalent to the system (1) for an *arbitrary* function f.

It is well known that (1) is integrable for $f=a(t)x+b(t)$, where a and b are arbitrary functions [18]. In that case the Lax pair is given by ([19], see also [13, 16]):

$$\Psi_{,x}=\begin{pmatrix}i\lambda & q\\ -\bar{q} & -i\lambda\end{pmatrix}\Psi \tag{3a}$$

$$\Psi_{,t}=\begin{pmatrix}iR-2if\lambda^2 & i(fq)_{,x}-2\lambda qf\\ i(f\bar{q})_{,x}+2\lambda\bar{q}f & 2if\lambda^2-iR\end{pmatrix}\Psi \tag{3b}$$

where

$$\lambda=\frac{k}{2}\left(1+k\int^t a(\tau)\,\mathrm{d}\tau\right)^{-1} \tag{3c}$$

and, finally, k is a constant: the spectral parameter.

If one prefers to call λ a spectral parameter then the linear system (3) is an example of a 'non-isospectral' Lax pair [20].

The coupling functions different than $f=ax+b$ seem to define non-integrable systems ([17, 21], see also [20]). The algorithm of [6] applied to (1) isolates $f=a(t)x+b(t)$ as an integrable case but a and b have to satisfy the following constraint:

$$\left(b(t)-c_1a(t)-c_2a(t)\int^t a(\tau)\,\mathrm{d}\tau\right)a(t)=0 \tag{4}$$

where c_1, c_2 are constants. Thus not *all* integrable cases are isolated in this way.

The aim of this letter is to show that by an appropriate extension of the algorithm of [6] one can isolate *all* coupling functions for which the NHNS system is known to be integrable.

In the standard approach one studies 1-parameter groups of Lie point symmetries of NHNS system leaving the coupling function f invariant. Our extension consists firstly in admitting symmetries transforming f as well, and secondly, in admitting a more general class of symmetries.

Let us consider the following 1-parameter family T_k of transformations acting in the infinite dimensional space parametrized by (x, t, q, R, a, b), where $x, t, R \in \mathbb{R}$, $q \in \mathbb{C}$ and a, b are elements of appropriately chosen function space (the convenient choice is, for instance $\mathscr{L}^1[-\infty, \infty]$):

$$T_k x = \frac{x}{(1-k\int_c^t a(\tau)\,d\tau)^2} + 2k\int_c^t \frac{b(\tau)\,d\tau}{(1-k\int_c^\tau a(\vartheta)\,d\vartheta)^3} \tag{5a}$$

$$T_k t = t \tag{5b}$$

$$T_k q = q\left(1-k\int_c^t a(\tau)\,d\tau\right)^2 \exp\left(\frac{ikx}{1-k\int_c^t a(\tau)\,d\tau} + ik^2\int_c^t \frac{b(\tau)\,d\tau}{(1-k\int_c^\tau a(\vartheta)\,d\vartheta)^2}\right) \tag{5c}$$

$$T_k R = R \tag{5d}$$

$$T_k a = \frac{a}{(1-k\int_c^t a(\tau)\,d\tau)^2} \tag{5e}$$

$$T_k b = \frac{b}{(1-k\int_c^t a(\tau)\,d\tau)^4} - \frac{2ka}{(1-k\int_c^t a(\tau)\,d\tau)^2}\int_c^t \frac{b(\tau)\,d\tau}{(1-k\int_c^\tau a(\vartheta)\,d\vartheta)^3} \tag{5f}$$

where k is a sufficiently small real parameter whose domain of definition depends on a point of the space defined above and c is another (less important) parameter of the transformation T_k.

One can prove (see [22]) the following propositions:

Proposition 1. The transformation T_k given by (5) transforms the NHNS system (1) with $f = ax + b$ into the NHNS system with $f = (T_{-k}a)x + (T_{-k}b)$.

Proposition 2. Transformations T_k form a 1-parameter local group (see [9]) of (non-local) transformations and k is an additive parameter:

$$T_k \circ T_1 = T_{k+1}. \tag{6}$$

Proposition 3. The transformation T_k complete with the gauge transformation

$$T_k\Psi = \exp\left(\frac{i}{2}\sigma_3\left(\frac{kx}{1-k\int_c^t a(\tau)\,d\tau} + k^2\int_c^t \frac{b(\tau)\,d\tau}{(1-k\int_c^\tau a(\vartheta)\,d\vartheta)^2}\right)\right) \tag{7}$$

maps the non-parametric Lax pair (2) into the standard Lax pair (3) of the NHNS system.

Therefore we came to the conclusion that there exists a 1-parameter group of non-local symmetries of (1) which inserts the spectral parameter into the 'non-parametric Lax pair' (2) for an *arbitrary* f linear in x.

The mathematical interpretation of the presented 1-parameter group seems to constitute quite an interesting problem. The infinitesimal version of the transformation (5) is given by:

$$(\mathrm{d}/\mathrm{d}k)(T_k x)|_{k=0} = 2x \int_c^t a(\tau)\,\mathrm{d}\tau + 2\int_c^t b(\tau)\,\mathrm{d}\tau \tag{8a}$$

$$(\mathrm{d}/\mathrm{d}k)(T_k q)|_{k=0} = q\left(ix - 2\int_c^t a(\tau)\,\mathrm{d}\tau\right) \tag{8b}$$

$$(\mathrm{d}/\mathrm{d}k)(T_k a)|_{k=0} = 2a \int_c^t a(\tau)\,\mathrm{d}\tau \tag{8c}$$

$$(\mathrm{d}/\mathrm{d}k)(T_k b)|_{k=0} = 4b \int_c^t a(\tau)\,\mathrm{d}\tau - 2a\int_c^t b(\tau)\,\mathrm{d}\tau. \tag{8d}$$

Treating a and b as real variables one can hope to identify the transformation (8) as a higher symmetry of some covering equations [23]. In this context special attention should be given to equations ($5e, f$) which form the global version of the purely non-local transformation ($8c, d$).

By virtue of the presented propositions one can easily see that the original criterion of Levi, Sym and Tu [6] should be extended by admitting a more general class of symmetries (including some *non-local* transformations) and this is the most important conclusion of this letter. An attempt to define an improved algorithm to isolate integrable geometries is made in forthcoming papers [10, 17].

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